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INFINITE FREQUENCY SOLUTIONS TO DISCONTINUOUS CONTROL SYSTEMS WITH VARIABLE DELAY

BY

JULIO G. DIX

Department of Mathematics, Southwest Texas State University
San Marcos, TX 78666-4616, USA
e-mail: jd01@swt.edu

ABSTRACT

In this article, we study the frequency of zeros in the solution to $u'(t) = f(t, u(t-\tau(t)))$, where τ is a delay that depends on time, and f is a discontinuous function. First we show examples of solutions that have infinite frequency for autonomous systems with variable delay, and for non-autonomous systems with constant delay. Then we prove that infinite frequency solutions cannot come from finite frequency data. Also we prove that under certain conditions on the delay, the zero solution is the only solution that has infinitely many zeros in each interval of a fixed length.

1. Introduction

Delay equations, also known as retarded equations, come from differential equations in which the rate of change of a function depends on its past values. For instance, in population models in which individuals reproduce only after maturity, the population changes at a rate that depends on past values of population. Also, in mechanical and electrical control systems that react to a delayed feedback, the rate of change depends on previous states. For a general overview on delay equations, see e.g. Schmitt [4], Hale [2], Hale and Lunel [3], Diekmann et al. [1] and Utkin [7]. This last reference studies delay equations in which right the hand side is discontinuous.

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The objective of this article is to determine conditions under which the following equation does not admit infinite-frequency solutions:

$$
\begin{aligned}
u'(t) &= f\left(t, u(t-\tau(t))\right) \quad \text{for } t>0\,, \\
u(t) &= \phi(t) \quad \text{for } -a \le t \le 0\,,
\end{aligned}
\tag{1}
$$

where $\tau(t)$ represents a delay that satisfies $\tau_0 \le \tau(t) \le t+a$, for given positive constants τ_0, a. The initial datum ϕ is a function continuous on $[-a, 0]$, with $\phi(0)=0$. The function $f(t,u)$ is discontinuous at $u=0$; however, for $t \ge 0$, $u \ne 0$ it is bounded, continuous and satisfies $uf(t,u) < 0$. The last assumption on f provides systems with a negative feedback mechanism that makes $u(t)$ decrease when $u(t-\tau)$ is positive, and makes $u(t)$ increase when $u(t-\tau)$ is negative. Negative feedback systems yield oscillatory solutions that have been classified as slowly oscillating if there exists a time from which the zeros of the solution are at least one unit apart, and as rapidly oscillating, otherwise. Chapters XV and XVI in [1], and [8] present a systematic study and a good set of references for negative feedback systems.

The plan for this article is as follows. In §2 we show that solutions with infinite frequency exist, by showing two examples of equations that have solutions with infinitely many oscillations on every unit interval. In §3, we show that finite frequency data do not lead to infinite frequencies. Also in §3 we show that under certain conditions, solutions cannot have infinite frequency. This is done by showing that if a solution has infinitely many zeros in each unit interval, then this solution must be identically zero. By considering variable delay here, we extend Shustin's results in [5].

2. Existence of solutions

Solutions are obtained by integrating in consecutive time intervals. Let $T_0 = 0$, and $[T_0, T_1]$ be the largest interval on which $t - \tau(t) \le 0$. Then a continuous, unique solution on $[T_0, T_1]$ is given by

$$u(t) = \int_0^t f(\phi(s-\tau(s)))ds.$$

Note that $T_1 - T_0 \ge \tau_0 > 0$ because $\tau(.) \ge \tau_0$. Let $[T_1, T_2]$ be the largest interval in which $t - \tau(t) \le T_1$. Then the solution on $[T_1, T_2]$ is given by

$$u(t) = u(T_1) + \int_{T_1}^t f(u(s-\tau(s)))ds.$$

Again we have $T_2 - T_1 \geq \tau_0$. By repeating this integration step we obtain a continuous solution for all $t \geq 0$.

As an illustration of systems with negative feedback we have the following two examples. In the first example the system is autonomous, f depends on t only through u, and has variable delay. In the second example the system is non-autonomous, and has a constant delay. In both examples the solutions have infinitely many oscillations on unit intervals, and both are 2-periodic.

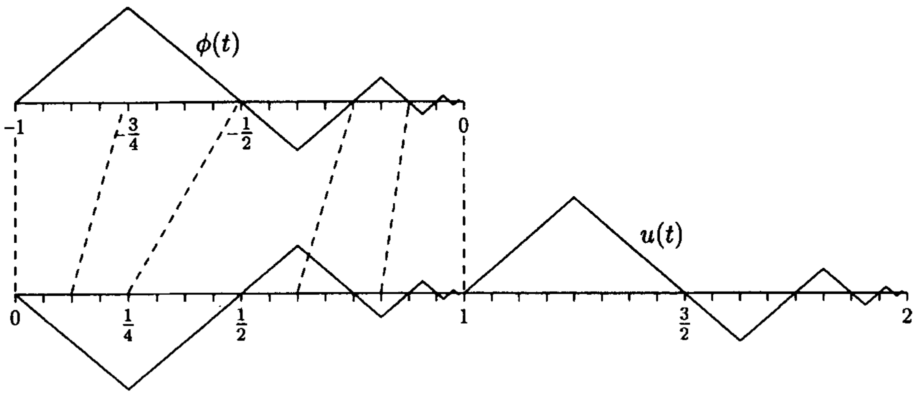

Figure 1. A solution with infinite frequency.

Example 1: Consider the delay equation

$$
\begin{aligned}
u'(t) &= -\text{signum}\, u(t - \tau(t)) \quad \text{for } t > 0, \\
u(t) &= \phi(t) \quad \text{for } -1 \leq t \leq 0,
\end{aligned}
$$

where ϕ is the linear interpolant through the points $(-1, 0)$, $(-3/2^i, (-1/2)^i)$, for $i = 2, 3, \ldots$. The delay τ is the linear interpolant through the points $(0, 1)$, $(1 - 3/2^i, 1 - 1/2^i)$, for $i = 2, 3, \ldots$; and τ is 1-periodic: $\tau(t) = \tau(t-1)$.

The graph of the solution is easily constructed from the graph of ϕ. Recall that $u(t)$ decreases with slope -1 when $u(t - \tau)$ is positive, and $u(t)$ increases with slope 1 when $u(t - \tau)$ is negative. See Fig. 1, where the mapping $t \mapsto t - \tau(t)$ is indicated with broken lines. For example, $t = 0$ is mapped to -1, $1/8$ is mapped to $-3/4$, and $1/4$ is mapped to $-1/2$.

Example 2: Consider the delay equation

$$
\begin{aligned}
u'(t) &= -\text{signum}\, u(t - 1) + p(t) \quad \text{for } t > 0, \\
u(t) &= \phi(t) \quad \text{for } -1 \leq t \leq 0,
\end{aligned}
$$

where ϕ is the linear interpolant through the points $(-1+3/2^i, -(-1/2)^i)$ and $(-3/2^i, (-1/2)^i)$, for $i = 3, 4, \ldots$.

The function p is defined as follows: $p(0) = 0$; $p(t) = 1/3$ on intervals of the form $(3/4^i, 6/4^i]$, $(1/2, 5/8]$, $(1-3/4^i, 1-/(4^i 2)]$ for $i = 2, 3, \ldots$; $p(t) = -1/3$ on intervals of the form $(3/(4^i 2), 3/4^i)$, $(3/8, 1/2]$, $(1-6/4^i, 1-3/4^i]$ for $i = 2, 3, \ldots$; $p(t) = 0$, for $1 < t \leq 2$; and p is 2-periodic so that $p(t) = p(t-2)$. See Fig 2.

To calculate the solution on $[0, 1]$, we use $u(t) = \int_0^t u'(s)ds$, where

$$\begin{aligned} u'(t) &= 1 - 1/3 \text{ on intervals } (2/4^i, 3/4^i), (1-6/4^i, 1-4/4^i), \\ u'(t) &= 1 + 1/3 \text{ on intervals } (3/4^i, 4/4^i), (1-8/4^i, 1-6/4^i), \\ u'(t) &= -1 + 1/3 \text{ on intervals } (4/4^i, 6/4^i), (1-4/4^i, 1-3/4^i), \\ u'(t) &= -1 - 1/3 \text{ on intervals } (6/4^i, 8/4^i), (1-3/4^i, 1-2/4^i), \end{aligned}$$

for $i = 2, 3, 4, \ldots$. Then the integral that defines u becomes a series. For example,

$$\begin{aligned} u(1/2) =& \tfrac{1}{8}(-1-\tfrac{1}{3}) + \tfrac{1}{8}(-1+\tfrac{1}{3}) + \tfrac{1}{16}(1+\tfrac{1}{3}) + \tfrac{1}{16}(1-\tfrac{1}{3}) \\ &+ \tfrac{1}{32}(-1+\tfrac{1}{3}) + \tfrac{1}{32}(-1-\tfrac{1}{3}) + \cdots. \end{aligned}$$

After some algebra we obtain $u(1/2) = -1/6$.

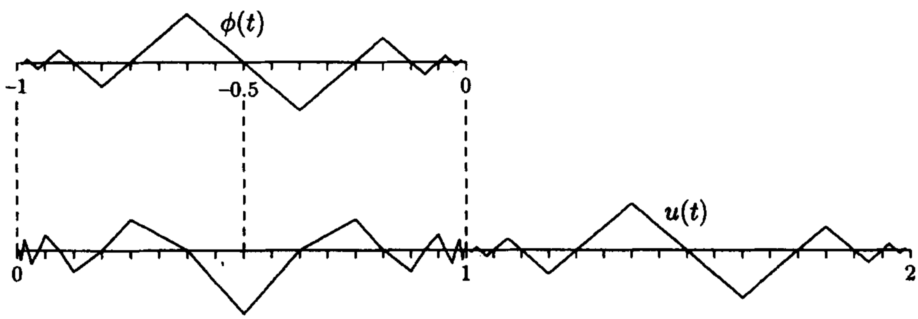

Figure 2. A solution with infinite frequency.

3. Frequency of oscillations

The next hypotheses preclude solutions having infinitely many oscillations, when the datum does not have infinitely many oscillations.

(A1) The only solution to $f(t, u) = 0$ is $u = 0$.

(A2) The mapping $s \mapsto s - \tau(s)$ is strictly increasing.

The "frequency" of a solution is measured by the number of zeros of the solution. Let $N(t)$ be the number of zeros of u in the interval $(t - \tau_0, t)$; and let

$\eta(t)$ be the number of zeros in the interval $(t^* - \tau(t^*), t^*)$, where t^* is the largest zero of u such that $t^* \leq t$.

LEMMA 1: *Assume* (A1), *and let* $t_1 < t_2$ *be consecutive zeros of* u *in some interval* $[T_i, T_{i+1}]$. *Then there is a value* s *such that* $u(s - \tau(s)) = 0$, *with* $t_1 < s < t_2$.

Proof: Suppose that there is no such value s; then the continuous function u does not change sign on (t_1, t_2). Hence $f(u(t - \tau(t)))$ is continuous on (t_1, t_2), and u is differentiable on (t_1, t_2). Since $u(t_1) = u(t_2) = 0$, by the Mean Value Theorem there exists s in (t_1, t_2) such that $u'(s) = 0$. Using (A1), we get $u(s - \tau(s)) = 0$, a contradiction that completes this proof. ∎

LEMMA 2: *Under assumptions* (A1–A2), $\eta(t)$ *is a non-increasing function.*

Proof: Let $t_1 < t_2$ be consecutive zeros of u in some interval $[T_i, T_{i+1}]$. Then $\eta(t_1)$ is the number of zeros in $(t_1 - \tau(t_1))$, and $\eta(t_2)$ is the number of zeros in $(t_2 - \tau(t_2))$. Because of (A2) there are only two possible cases.

Case $t_1 \leq t_2 - \tau(t_2) < t_2$. Since there are no zeros in the interval (t_1, t_2) it follows that $\eta(t_2) = 0$, and $\eta(t_2) \leq \eta(t_1)$.

Case $t_1 - \tau(t_1) < t_2 - \tau(t_2) < t_1 < t_2$. All zeros in $(t_1 - \tau(t_1), t_2 - \tau(t_2)]$ count for $\eta(t_1)$, but do not count for $\eta(t_2)$. By Lemma 1 there is at least one zero in this interval. Any zeros in $(t_2 - \tau(t_2), t_1)$ count for both $\eta(t_1)$ and $\eta(t_2)$. The only extra zero to be counted is t_1 which counts for $\eta(t_2)$, but not for $\eta(t_1)$. In conclusion $\eta(t_2) \leq \eta(t_1)$.

Applying the above argument to each pair of zeros of u we conclude that $\eta(t)$ is a non-increasing function. ∎

LEMMA 3: *Under assumptions* (A1–A2), *the frequency of zeros* $N(t)$ *is dominated by the non-increasing function* $\eta(t) + 1$, *in the sense that*

$$N(t) \leq \eta(t) + 1.$$

Proof: As in the definition of $\eta(t)$, let t^* be the largest zero of u such that $t^* \leq t$. Then consider the two possible cases.

Case $t^* \leq t - \tau_0 < t$. Since $N(t)$ is the number of zeros in $(t - \tau_0, t)$, $N(t) = 0$ and $N(t) \leq \eta(t)$.

Case $t - \tau_0 < t^* \leq t$. Since $\tau_0 \leq \tau(.)$, it follows that $(t^* - \tau_0, t^*)$ is a subset of $(t^* - \tau(t^*), t^*)$; therefore, $N(t^*) \leq \eta(t^*)$. Since the only zero in $[t^*, t)$ is t^*, and $t - \tau_0 < t^*$, we have $N(t) \leq N(t^*) + 1$. Combine the last two inequalities and the definition of t^* to get

$$N(t) - 1 \le N(t^*) \le \eta(t^*) = \eta(t). \qquad \blacksquare$$

The next theorem follows from this lemma and the definition of $N(t)$.

THEOREM 1: *Under assumptions* (A1–A2), *if the datum ϕ has finitely many zeros, then the solution has finitely many zeros in every bounded interval.*

For the remainder of this article we restrict our attention to the equation

$$\text{(2)} \qquad \begin{aligned} u'(t) &= \begin{cases} -p_0 & \text{if } u(t-\tau(t)) > 0, \\ 0 & \text{if } u(t-\tau(t)) = 0, \\ p_1 & \text{if } u(t-\tau(t)) < 0, \end{cases} \\ u(t) &= \phi(t) \qquad \text{for } -a \le t \le 0, \end{aligned}$$

where p_0, p_1 are positive constants, and the delay $\tau(t)$ is a non-decreasing function that satisfies (A2).

Following Shustin's idea [5], to each solution state we associate a sequence consisting of the length of the intervals where the solution is non-zero. Also, we define an operator that represents going back one time-step in the solution states. A sequence $\{x_n\}$ is said to be **summable** if $\|X\|_1 = \sum |x_n| < \infty$, and to be **bounded** if $\|X\|_\infty = \sup |x_n| < \infty$. Given a constant $0 < \alpha < 1$, define B as the operator that, to summable sequences $X = \{x_n\}_{-\infty}^{+\infty}$ of non-negative numbers, assigns the sequence

$$BX = \{x_n^1\}, \qquad \text{where } x_n^1 = \alpha x_n + (1-\alpha)x_{n+1}.$$

Note that x_n^1 is a convex combination of x_n and x_{n+1}, and that $\|X\|_1 = \|BX\|_1$.

LEMMA 4: *For positive constants τ_1, the repeated application of the operator B satisfies*

$$\lim_{k\to\infty} \sup_{\sum x_n \le \tau_1} \|B^k X\|_\infty = 0.$$

Proof: Using the notation $\binom{k}{i} = k!/(i! \cdot (k-i)!)$, it is not difficult to show by induction that the terms of $B^k X$ are

$$x_n^k = \sum_{i=0}^{k} \binom{k}{i} (1-\alpha)^{k-i} \alpha^i x_{n+i}.$$

Since B is a linear operator, we divide by $\sum x_n$ so that we may assume without loss of generality that $\sum x_n = 1$. Then x_n^k is a convex combination with weights

x_{n+i}. Being a convex combination, x_n^k is bounded above by the largest coefficient in the above summation; i.e.,

$$x_n^k \leq a_k := \binom{k}{i_k}(1-\alpha)^{k-i_k}\alpha^{i_k},$$

where i_k is an integer that yields the largest coefficient. Since these coefficients form a binomial distribution whose probability density function is concave, the maximum is attained at $i_k = \lfloor (k+1)\alpha \rfloor$, the largest integer i for which

$$\binom{k}{i-1}(1-\alpha)^{k-(i-1)}\alpha^{i-1} \leq \binom{k}{i}(1-\alpha)^{k-i}\alpha^{i}.$$

Because a_k is independent of X and of n, by showing that the non-negative sequence $\{a_k\}$ approaches zero as k approaches infinity, we conclude the statement of this lemma. To show that a_k does not increase, we show that the ratio a_{k+1}/a_k is bounded by one. Then by finding a subsequence that converges to zero, we conclude that $\{a_k\}$ converges to zero.

To show that $a_{k+1}/a_k \leq 1$, we consider the two possible cases:

CASE $i_{k+1} = i_k + 1$: Then there is an integer between $(k+1)\alpha$ and $(k+2)\alpha$, but this integer cannot be $(k+1)\alpha$ because $\alpha < 1$. In this case,

$$\frac{a_{k+1}}{a_k} = \frac{\binom{k+1}{i_k+1}}{\binom{k}{i_k}}\alpha = \frac{(k+1)\alpha}{i_k+1} = 1 - \frac{\lfloor (k+1)\alpha \rfloor + 1 - (k+1)\alpha}{i_k+1} \tag{3}$$

which is strictly less than 1.

CASE $i_{k+1} = i_k$: Then $(k+1)\alpha$ is the only possible integer between $(k+1)\alpha$ and $(k+2)\alpha$. In this case,

$$\frac{a_{k+1}}{a_k} = \frac{\binom{k+1}{i_k}}{\binom{k}{i_k}}(1-\alpha) = \frac{(k+1)(1-\alpha)}{k+1-i_k} = 1 - \frac{(k+1)\alpha - \lfloor (k+1)\alpha \rfloor}{k+1-i_k}. \tag{4}$$

Since $\lfloor (k+1)\alpha \rfloor \leq (k+1)\alpha$, the right hand side is less than or equal to 1. In summary, these two cases show that $\{a_k\}$ does not increase.

Among the integers k that satisfy $i_{k+1} = i_k + 1$, we shall find an increasing sequence $\{k_j\}$ such that $\{a_{k_j}\}$ converges to zero.

If the numerator of the right-most term in (3) converges to zero, then for each $\epsilon > 0$ there exists an integer k_0 such that

$$\lfloor (k+1)\alpha \rfloor + 1 - (k+1)\alpha < \epsilon$$

for all $k \geq k_0$, for which $\lfloor (k+1)\alpha \rfloor + 1 = \lfloor (k+2)\alpha \rfloor$. Then there is an integer between $(k+1)\alpha$ and $(k+2)\alpha$. Because $0 < \alpha < 1$, $\epsilon = (1-\alpha)/2$ is positive and there is no integer between $k\alpha$ and $(k+1)\alpha$, and

$$\frac{1-\alpha}{2} < \lfloor k\alpha \rfloor - k\alpha.$$

Since $\lfloor k\alpha \rfloor = \lfloor (k+1)\alpha \rfloor$, we apply inequality (4), and use the facts that: $\{a_k\}$ does not increase, $k_{j-1} \leq k_j - 1$, and $i_k + 1 \leq k$ to get

$$a_{k_j} \leq \left(1 - \frac{1-\alpha}{2k_j}\right) a_{k_{j-1}}.$$

If the numerator of the right-most term in (3) does not converge to zero, then there is a constant β and an increasing sequence $\{k_j\}$ such that

$$\lfloor (k_j+1)\alpha \rfloor + 1 - (k_j+1)\alpha \geq \beta > 0.$$

Then using the facts that: $\{a_k\}$ does not increase, $k_{j+1} \geq k_j + 1$, and $i_k + 1 \leq k$, from (3) we get

$$a_{k_{j+1}} \leq a_{k_j+1} \leq \left(1 - \frac{\beta}{k_j}\right) a_{k_j}.$$

From the two cases above, for the positive constant $\lambda = \min\{(1-\alpha)/2, \beta\}$ there exists a subsequence $\{a_{k_j}\}$ that satisfies $a_{k_{j+1}} \leq (1 - \lambda/k_j) a_{k_j}$. Repeated applications of this inequality yield

$$a_{k_{j+1}} \leq a_{k_1} \prod_{i=1}^{j} \left(1 - \frac{\lambda}{k_i}\right).$$

The right hand side of this inequality approaches zero because its logarithm approaches $-\infty$. In fact, the Taylor expansion $\log(1-x) = -x - x^2/2 - x^3/3 - \cdots$ for $0 \leq x < 1$ yields an upper bound for the logarithm of the product above:

$$\log \prod_{i=1}^{j} \left(1 - \frac{\lambda}{k_i}\right) \leq -\lambda \sum_{i=1}^{j} \frac{1}{k_i}$$

which diverges to $-\infty$. In conclusion, a subsequence $\{a_{k_j}\}$ approaches zero. Hence the sequence $\{a_k\}$ approaches zero; which completes this proof. ∎

Remark 1: The iterates $B^k X$ converge to zero at a rate no faster than $k^{-1/2}$. From the last inequality in the proof of Lemma 4

$$\log(a_k) \leq \log(a_1) - \lambda \sum_{i=1}^{k} \frac{1}{i} \approx -\lambda \log(k).$$

Thus a_k is asymptotically equivalent to $k^{-\lambda}$. Then the largest possible $\lambda = 1/2$ marks the fastest rate of convergence. For the constant delay case, the optimal rate was shown in [5]. There the proof uses estimates on the area between the graph of u and the t-axis, rather than the supremum norm used here.

Next we use the operator B in building an estimate for the length of intervals where the solution is not zero. Let $w < z$ be zeros of u in an interval $[T_i, T_{i+1}]$, such that the interval (w, z) does not have accumulation points of zeros of u. To shorten notation we use u^i to indicate the solution u restricted to the interval $(w, z) \subset [T_i, T_{i+1}]$.

Enumerate consecutively the intervals where u^i is non-zero. Use odd indices when u is positive, even indices when u is negative, and skip an index if u^i has the same sign in two consecutive intervals. Denote the length of these intervals as $\{u_n^i\}_{-\infty}^{\infty}$, with $u_n^i = 0$ if the n-th interval does not exist.

The graph of u^i is piecewise linear, and zig-zags around the t-axis, but not all lines cross the t-axis. We eliminate lines that do not reach the t-axis by prolonging neighboring lines that reach the t-axis; see Fig. 3. Note that the modified state v^i and the original u^i generate the same solution state u^{i+1}.

Apply the mapping $t \mapsto t - \tau(t)$ to the inflection points of v^i, to obtain points $\{s_j\}$ in $[T_{i-1}, T_i]$. Through the points $\{s_j\}$ build a graph v^{i-1} that is piecewise linear with slopes $-p_0$ and p_1, and that as datum for the delay equation generates v^i. Put

$$v_j^{i-1} = s_{j+1} - s_j,$$

so that for non-decreasing τ, by setting $\alpha = p_0/(p_0 + p_1)$, we have

$$v_{2n-1}^{i-1} \leq (u_{2n-1}^i + u_{2n}^i)\alpha, \quad v_{2n}^{i-1} \leq (u_{2n}^i + u_{2n+1}^i)(1 - \alpha).$$

From Fig. 3, where τ is assumed constant, it can be seen that the above inequalities then become equalities.

From this construction we observe that: v^{i-1} and u^{i-1}, as initial datum functions, generate the same solution state u^{i+1}; on intervals where u^i is not modified, v^{i-1} and u^{i-1} have the same zeros; v^{i-1} does not have more zeros than u^{i-1}; and $\|u_n^{i-1}\|_\infty \leq \|v_n^{i-1}\|_\infty$.

Next we establish the relationship between the operator B and the construction of v^{i-1}. By adding pairs of entries we build two new sequences X and X^1 whose entries are:

$$x_{2n-1} = u_{2n-1}^i + u_{2n}^i \quad \text{and} \quad x_{2n-1}^1 = u_{2n-1}^{i-1} + u_{2n}^{i-1}.$$

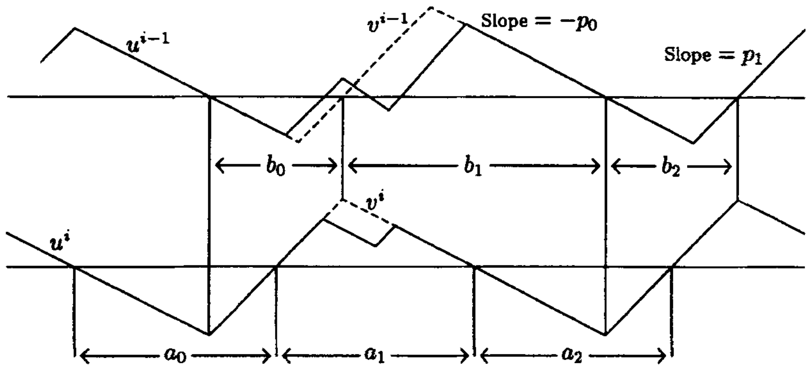

Figure 3. Modified state v^i.

We observe that BX as defined before coincides with X^1, and that $\sup\{u_n^{i-1}\} \leq \sup\{v_n^{i-1}\} \leq \|BX\|_\infty$. Repeated applications of the above procedure yield states v^{i-k}, such that

(5) $$\sup\{u_n^{i-k}\} \leq \sup\{v_n^{i-k}\} \leq \|B^k X\|.$$

All the background has been set for the main theorem in this article.

THEOREM 2: *Suppose that the delay τ is a non-decreasing function that satisfies $0 < \tau_0 \leq \tau(t) \leq \tau_1$, and that the mapping $t \mapsto t - \tau(t)$ is invertible. If a solution to* (2) *has infinitely many zeros in each unit interval, then this solution is identically zero.*

Proof: Let u be the solution to (2) and let Z be the set of zeros of u. Let $[T_i, T_{i+1}]$ be the intervals used in the construction of u from the integral form of (2). Then $\tau_0 \leq T_{i+1} - T_i \leq \tau_1$.

Since every unit interval has infinitely many zeros of u, for each interval $[T_i, T_{i+1}]$ there is $i_0 > i$ such that $[T_{i_0}, T_{i_0+1}]$ contains infinitely many zeros. Therefore, by Lemma 1, $[T_i, T_{i+1}]$ contains infinitely many zeros. Hence, by the Heine–Borel Theorem each $[T_i, T_{i+1}]$ contains accumulation points of Z.

Assume that u is not identically zero. Then there are accumulation points w_0, z_0 such that w_0 is in $[T_0, T_1]$, and the interval (w_0, z_0) does not have accumulations points of Z. Let u^0 denote u restricted to the interval $[w_0, z_0]$. Let $g(.)$ denote the inverse of the mapping $t \mapsto t - \tau(t)$.

Let w_1, z_1 be accumulation points of Z, such that (w_1, z_1) is the largest interval that contains $(g(w_0), g(z_0))$, but does not contain any accumulation points of Z. This interval exists because every unit interval has accumulations points of Z, and Lemma 1 does not allow any accumulation points in $(g(w_0), g(z_0))$. The length of (w_1, z_1) does not exceed τ_1, because by Lemma 1, when t is an accumulation point so is $t-\tau(t)$. In the same manner we define the accumulation points w_k, z_k, such that the length of (w_k, z_k) does not exceed τ_1.

Following the procedure explained prior to this theorem, we build the solution estate v^k on the domain $[w_k, z_k]$, and the sequence X which satisfies $\|X\|_1 \leq \tau_1$. Then build the solution state v^{k-1} on the domain $[w_k - \tau(w_k), z_k - \tau(z_k)]$. Then build v^{k-2}, and so on. From the assumption of u not being identically zero and (5), we obtain that for all k,

$$0 < \|u^0\|_\infty \leq \|v^{k-k}\|_\infty \leq \|B^k X\|.$$

But this contradicts the convergence of $\|B^k X\|$ as stated in Lemma 4; therefore, u is identically zero on $[w_0, z_0]$. Since u is a continuous function that cannot have intervals without accumulation points of its zeros, it follows that u is identically zero. ∎

Remark 2: Solving for u in (1) is not a stable process. In the space of continuous functions, there exist initial functions ϕ_n that converge to a function ϕ, while the corresponding solutions do not converge to the solution generated by ϕ. Let $a = p_o = p_1 = 1 = \tau(t) = 1$ in (2). Then the solution is the 4-periodic function

$$u(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ -t, & \text{if } 1 \leq t < 3, \\ t, & \text{if } 3 \leq t < 4. \end{cases}$$

Each term of the sequence $\phi_n(t) = \frac{1}{n}\phi(t)$, $i = 1, 2, 3, \ldots$ generates the same solution $u(t)$. While the ϕ_n's converge to zero, the solutions do not converge to the zero solution. This proves instability at the zero solution, but the process may be stable at some other solutions since (2) is not a linear problem. Stability has been also interpreted in terms of limiting frequencies by considering the $\limsup_{t\to\infty} \eta(t)$. For this interpretation, some stability results can be found in [6].

Remark 3: In order to apply Theorem 2 to more general right-hand sides, Shustin [5] adds a perturbation of the form $|uf(u,t)| < \min\{p_0, p_1\}$, and then uses a change of variables

$$w = u + u^2 g(u,t),$$
$$s = t + u^2 h(u,t),$$

such that u and w have the same zeros. This change of variables leads to a variable delay $\tau(u,t)$ that oscillates slightly around a constant level. Although the same procedure applied here would expand our results, finding more general right-hand sides for which Theorem 2 applies is an open question.

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